What is Cantor's continuum problem?

on Kurt Gödel's article of the same title

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This seminar paper reviews Kurt Gödel's article "What is Cantor's continuum problem?"[1]. Some background in mathematics and logic is essential to understand Gödel's text. As this paper aims to be almost self-contained, short recaps, rough explanations and selective examples are provided where appropriate. Technical details are omitted in these parts though. It is structured in a way that the versed reader can easily skip the examples and "Properties of" sections without loosing the plot.

Section 0 frames the general context recalling relevant findings and terms in mathematical logic. Since Gödel's article is structured in a sensible way sections 1-5 proceed analog to Gödel's article. The first sections introduce cardinal numbers and the continuum hypothesis. Later sections bring up questions about the foundations of set theory and Gödel discusses possible solutions to the continuum problem. Section 6 closes with a discussion.

0. Context

Kurt Gödel's[2] paper entitled "What is Cantor's continuum problem?"[3] was first published in *The American Mathematical Monthly* in 1947[4]. At this time he was at the Institute for Advanced Study in Princeton, New Jersey. His completeness theorem as well as his incompleteness theorems were already proven years before[5]. Later, in 1983, a revised and expanded version of this paper was published in Paul Benacerraf[6] and Hilary Putnam's[7] "Philosophy of Mathematics: Selected Readings"[8].

At this opportunity the quintessence of Gödel's completeness and incompleteness theorems as well as some essential terms are rather informally recalled below.

[1] Göd83.
[2] Kurt Friedrich Gödel; * 1906; † 1978.
[3] Georg Ferdinand Ludwig Philipp Cantor; * 1845; † 1918.
[4] Göd47.
[5] Göd30; Göd31.
[6] Paul Benacerraf; * 1931.
[7] Hilary Whitehall Putnam; * 1926.
[8] Göd83.

0.1. Properties of first-order logic

soundness

$$\vdash P \quad \Longrightarrow \quad \models P$$

completeness theorem

$$\vdash P \quad \Longleftarrow \quad \models P$$

completeness

$$\vdash P \quad \Longleftrightarrow \quad \models P$$

compactness if a formula P follows from a (possibly infinite) set of formulas X, there exists a finite subset $X' \subseteq X$ such that P follows from this finite set of formulas

Löwenheim-Skolem theorem

downward infinite structures have elementary substructures of all smaller infinite cardinalities

upward infinite structures have elementary extensions of all larger cardinalities

0.2. Properties of theories $T \subseteq \mathcal{L}$

consistent a theory T is consistent *iff*

$$\nexists \varphi \in \mathcal{L} \; : \; \varphi \in T \text{ and } \neg\varphi \in T$$

(negation) complete a theory T is (negation) complete *iff*

$$\forall \varphi \in \mathcal{L} \; : \; \varphi \in T \text{ or } \neg\varphi \in T$$

independent sentence a sentence $\varphi \in \mathcal{L}$ is said to be independent of a theory T *iff* neither φ nor $\neg\varphi$ is provable from that theory (so the theory is incomplete)

incompleteness theorem I every *recursively axiomatized*[9] *sufficiently expressive*[10] (rase) theory can not be both consistent and complete;
every rase theory which is consistent is incomplete

incompleteness theorem II if a rase theory is consistent it can not prove its own consistency

[9] One can think of a theory being "recursively axiomatized" as "a computer program can recognize whether a given proposition is an axiom of that theory".

[10] One can think of "sufficiently expressive" as "capable of expressing all primitive recursive functions, properties and relations". Robinson[11] investigated the minimal prerequisites (Rob50).

[11] Raphael Mitchel Robinson; * 1911; † 1995.

0.3. Link to the continuum problem

Obviously Kurt Gödel's incompleteness theorems somehow settled David Hilbert's[12] second problem[13] and sabotaged Hilbert's Programme: arithmetic can not prove its own consistency.

Every *consistent* recursively axiomatized theory which contains Robinson arithmetic (RA) is incomplete and hence even incompletable. Furthermore it can not prove its own consistency. This applies to all the known theories like RA, Peano arithmetic (PA), Principia Mathematica and Zermelo-Fraenkel set theory with and without axiom of choice (ZF(C)).

Cantor's continuum problem, aka the continuum hypothesis (CH)[14], is again one of Hilbert's problems. It is linked to the aforementioned findings in the following way:

In 1938 Gödel showed that given consistency CH can not be disproved from ZF(C), i. e. ¬CH is not provable[15]. Nine years later he published the first version of his paper "What is Cantor's continuum problem?". When writing up the extended version of his article[16] Gödel got to know about Cohen's[17] result at the last moment and added a postscript: In 1963 Cohen showed that given consistency CH is not provable either[18].

In a nutshell: Assuming consistency of a rase[9,10] theory like ZF(C) there must be independent statements due to Gödel's incompleteness theorems. ZF(C) is widely assumed to be consistent and hence incomplete. It turned out that under this assumption CH is independent from ZF(C), i. e. neither provable nor disprovable from ZF(C).

1. The concept of cardinal number

At first Gödel rephrases Cantor's continuum problem as one of the following questions.

> "How many points are there on a straight line in euclidean space?"
>
> "How many different sets of integers do there exist?"

Obviously these rephrased questions do not answer the article's main question "What is Cantor's continuum problem?". Later it will become clear how they connect to CH.

As Gödel points out, these questions already depend on the concept of numbers being extended to infinite sets. To justify these simplified questions he introduces Cantor's definition of infinite numbers, the concept of cardinal numbers. He claims that –after closer examination– this will turn out to be *the* uniquely determined way of extending the concept of numbers.

[12] David Hilbert; * 1862; † 1943.
[13] "Prove that the axioms of arithmetic are consistent."
[14] The CH is a hypothesis of general mathematical interest which was advanced by Cantor in 1878. Below its statement will be clarified.
[15] Göd38.
[16] The revision was already part of the first edition of the selected readings published in 1964.
[17] Paul Joseph Cohen; * 1934; † 2007.
[18] Coh63.

He argues as follows: Which cardinal number is assigned to a set should not depend on the properties of the objects contained in that set. Gödel asserts that "we certainly want" that the number assigned to a set stays the same no matter how the properties or mutual relations of objects in that set are being changed. He refers to mapping the unit interval one-to-one on the unit square by changing properties of the "mass points" in the unit interval (cf. example 2). Supposedly, this supports his argument for sets of physical objects.

The concept of numbers is said to be satisfactory only if it is not dependent "on the kind of objects that are numbered" be they physical objects or not. Gödel concludes that "there is hardly any choice left but to accept Cantor's definition of equality between numbers"[19]. Therefore "we certainly want" that the extension of the concept of numbers to infinite sets preserves Cantor's definition of equality between numbers assigned to sets. ?[1]

Equal, greater and *less* are defined as usual, e. g. (cardinal) numbers are equal *iff* there is a bijection between the corresponding sets. Existence of a cardinal number is identified with existence of a set of that cardinal number. Since the power set of a set has always greater cardinality than the set itself (cf. example 1), these definitions yield infinitely many different cardinal numbers.

Gödel points out that "all ordinary rules of computation" and arithmetical operations can be extended to cardinal numbers "without any arbitrariness" as well. This way Gödel tries to pin down the concept of cardinal numbers as *the* unique extension of the concept of numbers.

In a final step Gödel employs the ordinary systematic representation of cardinal numbers which allows to uniquely identify the cardinal number belonging to an individual set. This is "to denote the cardinal number immediately succeeding the set of finite numbers by $\aleph_0$, the next one by $\aleph_1$, etc.", $\aleph_\omega$ immediately succeeds all $\aleph_i$ where $i \in \mathbb{N}$ and is followed by $\aleph_{\omega+1}$ and so forth.

1.1. Examples

Gödel indirectly referred to some of the following examples which demonstrate some properties of the concept of (infinite) numbers. Furthermore they are sometimes surprising when proving our intuition wrong or at least casting them into doubt. Having these examples ready helps understanding the (rephrased) statement of the continuum problem.

Example 1. (Cantor's theorem)
Let A be a set. Clearly $|A| \leq |\mathcal{P}(A)|$ since $x \mapsto \{x\}$ is an injective map $A \to \mathcal{P}(A)$. Assume there is a surjective map $f : A \to \mathcal{P}(A)$. Set $C := \{x \in A \ : \ x \notin f(x)\}$. Since f is assumed to be a surjective map $A \to \mathcal{P}(A)$ and $C \in \mathcal{P}(A)$ there exists $a \in A$ for which $f(a) = C$. $a \in f(a) = C \iff a \notin f(a) = C$ yields a contradiction. Therefore the assumption must be wrong and hence

$$|A| < |\mathcal{P}(A)| = 2^{|A|}.$$

[19]What Gödel refers to as "Cantor's definition of equality between numbers" is also often called "Hume's principle"[20].
[20]David Hume; * 1711; † 1776.

Example 2. (unit interval and unit square)
For every number x in the unit interval $[0,1) \subsetneq \mathbb{R}$ consider the decimal representation

$$x = \sum_{0<n\in\mathbb{N}} \frac{\alpha_n}{10^n}, \forall 0 < n \in \mathbb{N} : \alpha_n \in \{0,1,2,3,4,5,6,7,8,9\}.$$

Forbidding decimal representations with a trailing infinite sequence of 9s makes this representation unique while still allowing to represent every number x in the unit interval. Grouping the decimal places into groups which all range up to the next digit different from 9 yields a unique representation $x = 0, x_1 x_2 x_3 \dots$ (e. g. $0,3\ 990\ 4\ 97\ \dots$) for every $x \in [0,1)$ which enables the following definition:

$$\begin{aligned} f : [0,1) &\to [0,1) \times [0,1) \subsetneq \mathbb{R}^2 \\ 0, x_1 x_2 x_3 \dots &\mapsto (0, x_1 x_3 \dots ; 0, x_2 x_4 \dots) \end{aligned}$$

Since f is one-to-one, a bijection between the unit interval and the unit square is found; hence

$$|[0,1)| = |[0,1) \times [0,1)|.$$

Example 3. (open intervals of reals)
Let $a, b, c, d \in \mathbb{R}$ with $a < b$ and $c < d$. $(a,b) \subseteq \mathbb{R}$ and $(c,d) \subseteq \mathbb{R}$ have same cardinality since the following map is one-to-one.

$$\begin{aligned} f : (a,b) &\to (c,d) \\ x &\mapsto \tfrac{(x-a)}{(b-a)} \cdot (d-c) + c \end{aligned}$$

Especially $|(a,b)| = |(-\frac{\pi}{2}, \frac{\pi}{2})|$ for all $a, b \in \mathbb{R}$ with $a < b$. Since $\tan : (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$ is one-to-one it follows that

$$|(a,b)| = |\mathbb{R}| \text{ for all } a, b \in \mathbb{R} \text{ with } a < b.$$

Example 4. (reals and power set of naturals)
For every number x in the unit interval $[0,1] \subsetneq \mathbb{R}$ consider the binary representation

$$x = \sum_{n\in\mathbb{N}} \frac{\alpha_n}{2^{n+1}} = 0, \alpha_0 \alpha_1 \alpha_2 \dots, \forall n \in \mathbb{N} : \alpha_n \in \{0,1\}.$$

This allows the definition of the following one-to-one map.

$$\begin{aligned} f : [0,1] &\to \mathcal{P}(\mathbb{N}) \\ 0, \alpha_0 \alpha_1 \alpha_2 \dots &\mapsto \{n \in \mathbb{N} \ : \ \alpha_n = 1\} \in \mathcal{P}(\mathbb{N}) \end{aligned}$$

Hence

$$|[0,1]| = |\mathcal{P}(\mathbb{N})|.$$

1.2. Take-home message

Putting things together[21] yields

$$\aleph_0 = |\mathbb{N}| < |\mathcal{P}(\mathbb{N})| = {\scriptstyle |\mathbb{R}|=|(a,b)|=|\mathbb{R}^2|} = 2^{|\mathbb{N}|} = 2^{\aleph_0} \quad \text{where } a, b \in \mathbb{R} : a < b.$$

2. The continuum problem, the continuum hypothesis, and the partial results concerning its truth obtained so far

Introducing the concept of cardinal numbers gives meaning to the question "How many?". This allows Gödel to clarify the statement of CH with his vivid formulations above. The questions turn into identifying $|(a, b)|$ ("how many points on a straight line") or identifying $|\mathcal{P}(\mathbb{N})|$ ("how many sets of integers"). Indeed this is the same as identifying the number $|\mathbb{R}|$ of points on the *continuum*.

At this point Cantor's continuum hypothesis comes into play: Considering that $\aleph_0 = |\mathbb{N}| < |\mathbb{R}|$ Cantor conjectured that the cardinal number assigned to the continuum is $\aleph_1$.

CH $|\mathbb{R}| = \aleph_1$.

Hence supposing that there is no cardinal number between $|\mathbb{N}|$ and $|\mathbb{R}|$. Gödel puts this equivalently as "Any infinite subset of the continuum has the power either of the set of integers or of the whole continuum.".

The question CH tries to answer looks simple. Little preliminary work had to be done to finally come across this issue. Gödel stresses that nevertheless there has been no great breakthrough in answering CH. As is known, the cardinal number of the continuum is lower bounded by $\aleph_0 < |\mathbb{R}|$. It is undecided whether this number is regular or singular and whether it is accessible (cf. section 2.1. below for definitions). König's[22] theorem yields only a restriction for its cofinality.

In this section Gödel briefly demonstrates these shortcomings. He connects CH with general questions of cardinal arithmetic as CH can be rephrased as $2^{\aleph_0} = \aleph_1$. CH becomes a question of evaluating products and powers of cardinal numbers:

Some basic rules for multiplication and exponentiation of cardinal numbers are available. Assuming the generalized continuum hypothesis

GCH $2^{\aleph_\alpha} = \aleph_{\alpha+1}$ for every α.

all products and powers could more easily be evaluated (cf. example 6). This way he emphasizes the pronounced failure that there is no answer for (G)CH.

[21] Technical details have been omitted. E. g. to be more rigorous one would need to fix the inconsistent use of open, half-open and closed intervals in the examples.

[22] Gyula Kőnig; * 1849; † 1913.

2.1. Properties of cardinals and examples of cardinal arithmetic

cofinality the cofinality $\mathrm{cf}(m)$ of a cardinal number m is the smallest number n such that m is the sum of n numbers $< n$
(as a side note: one can derive $2^{\aleph_0} \neq \aleph_\omega$ using König's theorem – at cost of using the axiom of choice or postulating König's theorem itself)

regular a cardinal number m is regular *iff* $\mathrm{cf}(m) = m$

singular a cardinal number m is singular *iff* $\mathrm{cf}(m) \neq m$

(strongly) inaccessible an infinite cardinal number m is inaccessible if it is regular and has no immediate predecessor (i. e. m is a weak limit cardinal)
m is strongly inaccessible if each product of fewer than m numbers $< m$ is $< m$ (i. e. m is a strong limit cardinal)

Example 5. (cardinal exponentiation)
If $2 \leq \kappa \leq \lambda$ and λ is infinite, then $\kappa^\lambda = 2^\lambda$. This is proven by

$$2^\lambda \leq \kappa^\lambda \leq (2^\kappa)^\lambda = 2^{\kappa\lambda} = 2^\lambda$$

since ordinary rules of exponentiation hold. ($\kappa\lambda = \max(\kappa, \lambda) = \lambda$ assuming the axiom of choice.)

Example 6. (exponentiation assuming GCH)
Let $\aleph_\alpha$ be a regular cardinal number. Then $\mathrm{cf}(\aleph_\alpha) = \aleph_\alpha$ and hence ${\aleph_\alpha}^{\mathrm{cf}(\aleph_\alpha)} = {\aleph_\alpha}^{\aleph_\alpha}$. The previous example yields ${\aleph_\alpha}^{\mathrm{cf}(\aleph_\alpha)} = {\aleph_\alpha}^{\aleph_\alpha} = 2^{\aleph_\alpha}$. Assuming GCH then yields

$${\aleph_\alpha}^{\mathrm{cf}(\aleph_\alpha)} = {\aleph_\alpha}^{\aleph_\alpha} = 2^{\aleph_\alpha} = \aleph_{\alpha+1}.$$

3. Restatement of the problem on the basis of an analysis of the foundations of set theory and results obtained along these lines

The lack of results concerning CH might not be solely due to mathematical difficulties. Gödel raises the question of a more profound analysis of the used terms (e. g. set, one-to-one correspondence, etc.) and underlying axioms to resolve this "scarcity of results".

He points out that intuitionism as "certain philosophical conception" leads to destructive results in this field – e. g. rejecting all $\aleph$'s greater than $\aleph_1$ as meaningless. Gödel simplifies the key idea of this view, which only considers mathematical objects constructible from mathematical intuition.

In contrast, he sketches a somewhat more Platonistic view which allegedly allows to overcome this negative attitude towards Cantor's naive set theory:

Mathematical objects are considered to exist independently of construction or individual intuition. Talking about truth and soundness only requires the mathematical concepts to be "sufficiently clear".

If taking this viewpoint, Gödel *believes* that axiomatization of set theory can be interpreted in a way such that it serves as adequate foundation of set theory on its whole.

Paradoxes of naive set theory are dispelled as not causing mathematical trouble at all. The "perfectly naive and uncritical working" in mathematics with sets being obtainable from the set of integers by iterating the operation *set of*[23] is said to be proved self-consistent. Antinomies are avoided by not dividing the totality of all objects into two sets and by only forming sets on the basis of the set of integers. Hence Gödel views set-theoretical paradoxes as problem for logic and epistemology – if at all, since e. g. ZF(C) rules out the classic paradoxes of naive set theory.

He points out that the precise axiomatization of set theory allows transforming (almost) all mathematical proofs devised up to now, by means of mathematical logic, into a problem of manipulating symbols. Gödel makes a strong argument that "even the most radical intuitionist must acknowledge [this] as meaningful".

Gödel concludes that, no matter which philosophical standpoint is taken, Cantor's continuum problem retains the meaning of questioning whether and which answer of CH can be derived from the axiomatization in place. In any case CH is either provable, disprovable or independent if consistency is assumed. Gödel already showed that CH is not disprovable in that case.

Now comes a key point in Gödel's article. Taking the viewpoint stated above, mathematical entities exist in an objective way and the axioms try to capture this "well-determined reality". If CH turns out to be independent or in other words *undecidable*[24] still the question about the truth of CH in this reality remains unanswered. Gödel claims that CH must be either true or false in this reality. Accepting the axioms as sound he concludes that the axioms do not completely describe that reality if they can not decide CH. Gödel points out two possible ways of tackling this problem:

(a) On the one hand he states that the concept of set itself suggests that there must be axioms allowing still further iterations of the operation *set of*. He points to axioms about inaccessible cardinal numbers by Mahlo[25] allowing to apply the operation *set of* to the set of sets which one can build from the axioms already in place. This allows generating great transfinite cardinal numbers.

Gödel considers the extension of axiomatic set theory by this axiom as non-arbitrary: Since this great cardinal axiom "only unfold[s] the content of the concept of set" he argues that this extension is *implied* by the concept of set hence *necessary*. He seems to contradict himself when noting that ?2
the number of decidable propositions even "far outside the domain of very great transfinite numbers" increases when Mahlo's axioms are added – still they do not decide CH.

Gödel argues, that there could be axioms allowing a decision of CH for which there is what he calls an "intrinsic necessity".[26]

[23] Rationals can be seen as pairs of integers, reals as sets of rationals, real functions as sets of pairs of reals etc.

[24] Note again that Gödel got to know about Cohen's proof just after finishing the manuscript of this extended version of his article. Nevertheless he considered it to be very likely that CH is independent of ZF(C).

[25] Friedrich Paul Mahlo; * 1883; † 1971.

[26] In a footnote Gödel mentions an axiom which allows disproving GCH but lacks this intrinsic necessity.

(b) On the other hand he proposes a way towards the decision of CH which does not rely on agreeing about the *intrinsic necessity* of an axiom. It is about justifying a new axiom by its "success" or "fruitfulness". A new axiom is *fruitful* –and hence might be accepted– if it allows simpler and easier proofs of statements which also can be proven without adding this axiom. Gödel compares this to the verification of the axioms of the reals, rejected by intuitionists: They allow simpler proofs of theorems in number theory.

The last bit of this section contains a rather interesting link to physical theories: Gödel suspects that there might be such fruitful axioms, so fruitful that they simplify proofs, are rich of verifiable consequences and yield powerful (constructive) methods that –even if not intrinsically necessary– ?[3] "they would have to be accepted at least in the same sense as any well-established physical theory".

As his heading suggests, Gödel links Cantor's continuum problem to the foundations of set theory. The problem is viewed as a problem of the axiomatization of set theory. The question of provability or decidability from a certain set of axioms persists independently of the philosophical viewpoint. Taking the suggested Platonistic view also raises the question of finding new axioms which allow to decide CH *correctly* hence capture *more* of the "well-determined [mathematical] reality".

4. Some observations about the question: In what sense and in which direction may a solution of the continuum problem be expected?

Gödel expected what Cohen's proof finally confirmed: CH is independent hence undecidable from current set theory provided consistency. In this section he tries to give clear indications for his guess – but also somehow mixing it up with arguments for his claim that CH should be expected to be false in the *mathematical reality*. As it is known today that Gödel's guess (concerning the independence of CH) turned out to be right only arguments which are of further interest for later discussions are mentioned omitting mathematical details.

First he points out that current axioms do not contain the "characteristics of sets, namely, that they are extensions of definable properties" in the sense that this characteristics is neither explicitly nor implicitly captured by the axioms. In contrast to an axiom mentioned above which allows disproving CH[26], he points to another axiom which captures this definability property *and* allows to prove CH. This axiom states that every set is definable by ordinal numbers by certain procedures. It is consistent with the axioms already in place assuming their consistency. This also shows that adding CH to the axioms of set theory yields a consistent system, again assuming consistency of the axioms of set theory.

Putting things together: If CH can not be decided from current set theory there might be new axioms allowing for a decision[27], which might seem *intrinsically necessary* – e. g. one might argue

[27] Gödel of course already mentioned two of those possible axioms –the axiom of definability mentioned here and the

that the axiom of definability captures *more* of the mathematical reality and hence is *necessary.*

Gödel appeals to a rather pragmatic view, when listing "implausible consequences" of the continuum hypothesis. He gives examples of mathematical propositions which follow from CH but are supposedly *implausible.* Furthermore, without citing any example (!), Gödel points out that there are many *plausible* propositions which imply ¬CH while there is none which implies CH. "Adding up all that has been said" yields, according to Gödel, good reason for "suspecting that the role of the continuum problem in set theory will be to lead to the discovery of new axioms" which imply the negation of CH.

4.1. A sketchy roadmap of Gödel's line of argument so far

(i) take the Platonistic viewpoint that there is an independent mathematical reality

(ii) axiomatization of set theory tries to capture this reality

(iii) CH must be either true or false in this reality

(iv) but CH is neither provable[28] nor disprovable from the current axioms

(v) further axioms need to be discovered which allow a *correct* decision of CH

(vi) the assumption of CH leads to many implausible consequences

(vii) hence the axioms to be found need to allow to disprove CH

The last two points on this list are probably the most problematic ones since they rely on an interpretation of *plausibility.* In the next section Gödel supports these steps as "perfectly possible" by stating that the term *implausible* is not arbitrarily assigned to mathematical consequences or objects. He suggests the idea of a "mathematical intuition" about the mathematical reality which is said to be not purely subjective.

5. Supplement to the second edition

Gödel added the supplementary material to the extended version of his article to respond to new mathematical findings in connection with the continuum problem. De facto he only mentions three mathematical results and then elaborates quite a lot on Errera's[29] suggestion which leads him to a description of his idea of "mathematical intuition". He defends his standpoint that the question about the truth of CH is meaningful and important regardless of CH being independent of the axioms of set theory. This is in contrast to Errera who suggests that the question of the truth of

one mentioned in footnote 26– which do not lead to the same answer though.

[27] At least Gödel already assumed this to be the case. As already noted he was correct.

[29] Alfred Errera; * 1886; † 1960.

CH would then loose its meaning. He compares to the case of Euclid's parallel postulate where the question of truth lost its meaning by proving non-euclidean geometry consistent. Gödel argues that these two cases are very different and gives a mathematical and an epistemological reason.

(a) The mathematical argument goes as follows: Euclid's fifth postulate –let's call it E5 here– is independent of the other axioms of geometry –let's call them E– just like CH is independent of ZF(C). Now importantly, it occurs that *both* systems, one built from E and E5 the other from E and ¬E5, have a model in the original unextended system. Furthermore Gödel argues that both extensions are likewise *fruitful* in the sense explained above.

On the contrary, e. g. for the axiom of the existence of inaccessible cardinal numbers, there exists an asymmetry: only the system with the negation added has a model in the unextended system and contrariwise only asserting its truth yields a *fruitful* extension according to Gödel.

CH "can be shown sterile for number theory" and there is a model in the original system ZF(C)where CH is true. As this might not be the case when assuming other powers of the continuum, Gödel concludes that for CH there might be the same asymmetry which gives meaning to the question of the truth of CH – even if CH is independent from ZF(C).

(b) The epistemological argument leads Gödel also to an exposure of his idea of *mathematical intuition*. He argues that for a hypothetico-deductive system a question might loose meaning when proven undecidable. But as soon as the primitive terms in this system are "taken in a definite sense" the question retains its meaning. E. g. as soon as geometry refers to the behaviour of rigid bodies, the question of the parallel postulate retains its meaning.

He argues that this also applies to the question of the truth of CH in set theory. He admits though, that in geometry one does refer to physical objects or their sense perception whereas when giving meaning to the primitive set-theoretical terms one does not. The decision of truth of E5 in geometry is decided outside of mathematical intuition.

These two arguments try to reject Errera's suggestion and try to legitimate the last two steps of his line of argument presented in section 4.1. Gödel now works out his proposed concept of *mathematical reality* and *mathematical intuition*.

On the one hand the idea of *mathematical intuition* tries to retain the meaning of the question of the truth of CH. On the other hand Gödel's description of a *mathematical intuition* in mathematical practice might appeal to many mathematicians. Although set theory is remote of sense experience and does not belong to the physical world, Gödel assures that

> "we do have something like a perception also of the objects of set theory, as is seen from the fact that the axioms force themselves upon us as being true".

If this is the case is of course arguable. But Gödel concludes that this mathematical intuition can lead to the *correct* (cf. 4.1.(v)-(vii)) solution of Cantor's continuum problem just in the same way as physical theories are built up on sense perception.

It is expected that sense perceptions agree with physical theories and that (yet) undecidable questions have meaning in this setting and "may be decided in the future". Considering set-theoretical paradoxes as no more troublesome as deceptions of sense, Gödel argues that there is no reason why this kind of "mathematical perception" should be less trustworthy than sense perception.

Accepting this idea and taking terms of set theory in a definite sense retains the meaning of the question of the truth of CH comparably how physical interpretation does in the case of E5. Mathematical intuition about (the plausibility of) some implications of CH or its negation leading to a decision of CH is supposed to be "perfectly possible".

Gödel states that *mathematical intuition* should not be thought of giving immediate knowledge of the concerned objects. Instead the ideas of objects are formed based on *something* "which is immediately given". He compares this to physical experience where the sensations are immediately given. But ideas of physical objects contain "constituents qualitatively different from sensations" and still one can not just think of qualitatively new elements but only recombine given ones.

Although the objects of mathematical intuition can not be associated with our sense organs, Gödel claims that they are not purely subjective. They "may represent an aspect of objective reality", of *mathematical reality*, which may be *sensed* by us "due to another kind of relationship between ourselves and reality" as opposed to sense organs.

This "relationship between ourselves and reality" is not further explained by Gödel and might seem quite dubious. Additionally Gödel also tries to make his argument without requiring acceptance of his vaguely presented idea of "the objective existence of the objects of mathematical intuition". The sole existence of a clear intuition allowing to produce axioms of set theory is, as a psychological fact, said to be enough to give meaning to primitive elements of set theory and hence to the question of truth of CH.

Gödel closes that mathematical intuition could lead to an answer of CH. In contrast the presented criterion of fruitfulness (cf. section 3.(b)) can not *yet* settle the question of the truth of CH since too little is known about the consequences of CH. Number-theoretical consequences verifiable up to any given integer would easily allow to apply this criterion; but according to Gödel it is not yet possible to make the truth of any set-theoretical axiom "probable in this manner".

6. Discussion

Gödel answers the main question "What is Cantor's continuum problem?" by introducing cardinal numbers, rephrasing and explaining CH more figuratively and presenting the problem as a problem of the axiomatization of set theory. In addition he presents the idea of a mathematical reality of which one has a mathematical intuition which might lead to an answer to this axiomatization problem. Gödel claims that the problem should be fixed in a way such that finally CH can be disproved.

The first part, when explaining the continuum problem and how it relates to axiomatization of set theory, might seem absolutely watertight and mathematically rigorous. Here some assumptions are pointed out which sneak into Gödel's line of argument rather unnoticed. At first sight they might seem indubitable and therefore remained unnoticed.

Right at the beginning –when introducing the concept of cardinal numbers– there is a questionable argument. Gödel claims that "we certainly want" that the concept of numbers preserves the definition of equality by existence of a bijection. He refers to the intuitive idea that the number of elements in a set does not change if the properties of the elements are changed. Admittedly this seems rather sensible but still it is more of an intuitive argument. !1

Continuing with intuitive reasoning one could likewise expect that the concept of numbers allows to conclude $|(0,1)| < |\mathbb{R}|$ or $|(0,1)| < |(0,2)|$. So a straight line might intuitively be expected to be *smaller* than the whole real line. This connects to the examples which might be counterintuitive, e. g. it can be proved that there are as many points on the unit interval as on the unit square based on the concept of numbers Gödel presented (cf. example 2). E. g. measure theory tries to capture this intuition.

The demur is that there might be several *intuitive expectations towards the concept of numbers* which are not all fulfilled or fulfillable by the concept of numbers at the same time. Hence one would need to give good reason for the choice of intuitive expectations that are fulfilled by a certain concept of numbers.

To put it Gödelian: not all of the mathematical intuition about the mathematical reality of numbers might be captured by Cantor's concept of cardinal numbers. At this point it is especially critical since the concept of cardinal numbers might even contradict mathematical intuition.

After turning the problem into a problem of axiomatization of set theory Gödel concluded that the current axiomatization of set theory does not completely describe the mathematical reality. They do not allow for a decision of CH. When presenting two possible ways how one could get a complete description Gödel considers the possibility that there might be axioms which are intrinsically necessary, axioms which are *implied* by the concept of set. !2

Obviously Gödel contradicts himself, when assuming Mahlo's axiom about inaccessible cardinal numbers being of that kind and hence claiming that this axiom "only unfold[s] the content of the concept of set" while at the same time noting that this axiom allows for decision of many propositions outside the domain of cardinal numbers (e. g. Diophantine equations). So this axiom might just *not* only unfold the content of the concept of set but more than that.

There is another question one could raise here: Is there an axiomatization of set theory capturing all of mathematical reality? Assuming consistency of the axioms already in place there will always be independent/undecidable propositions due to Gödel's incompleteness theorem. If the axiomatization is extended to capture more of the mathematical reality there always will be further undecidable propositions[30]. Every undecidable proposition which can be formulated should be either true or false in the mathematical reality due to the same argument Gödel presents for CH being either true or false. Following this line of argument it is impossible to ever capture all of the mathematical reality Gödel proposes.

The aforementioned question connects to Gödel's comparison of fruitful axioms and useful physical theories. It seems like Gödel sees the connection between axioms and the mathematical reality !3 comparable to the connection between physical theories and the physical nature: both try to capture a *reality* which is objectively existent. So if an axiom is fruitful and yields powerful methods to deal with mathematical reality it needs to be accepted in the same way as physical theories are accepted if they help describing and predicting physical nature.

The mathematical reality then might be viewed in the same way as inaccessible objective existent reality as the physical nature which might also be inaccessible. It is interesting to further develop this analogy: There are well-established physical theories which are powerful in describing and predicting parts of physical nature but it turns out that they (a) not fully describe the physical nature (b) are incompatible with each other. Maybe there are powerful mathematical theories which show the same behaviour, maybe one can on the one hand describe much of mathematical reality with ZF(C)&CH and at the same time with ZF(C)&$\neg$CH?

However, Gödel's platonistic idea of mathematical reality is arguable. Still most mathematicians probably agree with him about a mathematical intuition. Mathematicians might well have an intuition about what their work *refers to* and what the *content* of their mathematical work is. Gödel even argues that mathematical intuition is not subjective. But it might be that much of the agreement about mathematical intuition –which hence seems to be objective– is due to being taught mathematical intuition from the cradle and one can not escape this predefined ways of intuitively thinking/doing mathematics.

Anyhow, one does not need to settle these questions before doing mathematics if taking the pragmatic view Gödel presented. If the axioms of set theory are accepted based on their fruitfulness it might be arguable how exactly to decide on the fruitfulness. Nevertheless pragmatism allows for a decision even if not eternally. Mathematical methods and the underlying axioms can be judged by their fruitfulness in descriptive disciplines like physics, they can be judged by mathematical elegance, simplicity or proof-power all on which *mathematicians* would need to agree.

Since it is impossible to eradicate all undecidable propositions CH might just be one easily accessible proposition raising such an agreement process. There might be consequences of CH or $\neg$CH which at one point will enforce the assumption of one or the other. Maybe this will then be as obvious as when people agreed upon the very basic and rarely doubted *modus ponens* or the axioms of ZF(C). It is to be noted, that the wide acceptance of ZF(C) is at its core also (only) based on agreement upon their fruitfulness, intuitive correctness and power for further mathematical activity.

[30]Note that every sentence becomes decidable if the extended theory is inconsistent.

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