UNIVERSITY OF COPENHAGEN

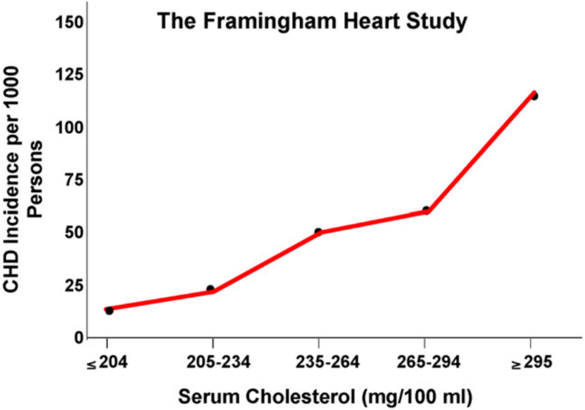
Causal Models under Variable Transformations

Challenges for *Causally Consistent* Representation Learning

Sebastian Weichwald

sweichwald.de @sweichwald

ETH Guest Lecture
2020-10-06



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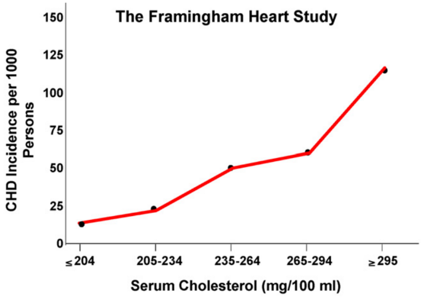

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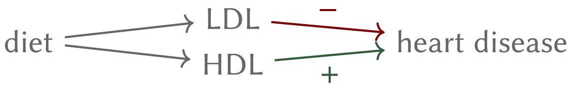
Variable Transformations may break Causal Reasoning ↯



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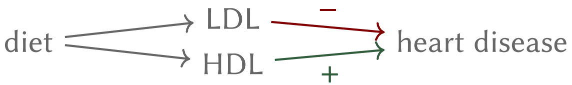

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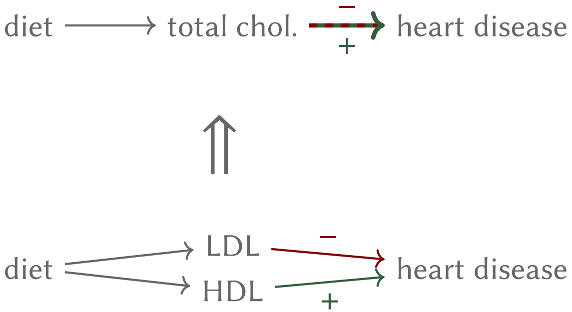
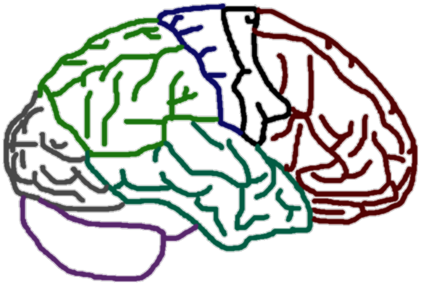


Rubenstein*, Weichwald*, et al (2017). Causal Consistency of Structural Equation Models. *Uncertainty in Artificial Intelligence.*

Observables may not be (meaningful) Causal Entities ↯



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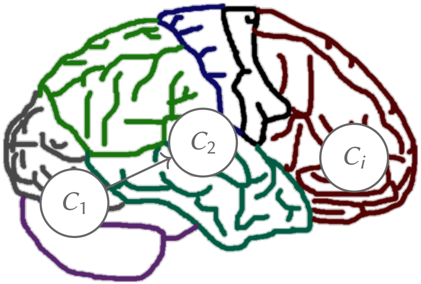

causal entities

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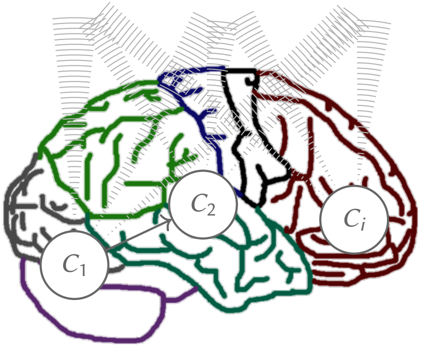

linear mixing

causal entities

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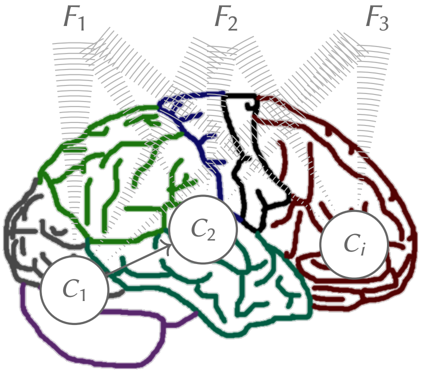

observed linear mixture

linear mixing

causal entities

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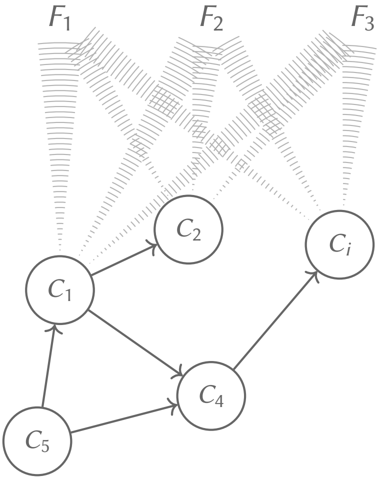


observed rgb pixel values

taking a photo

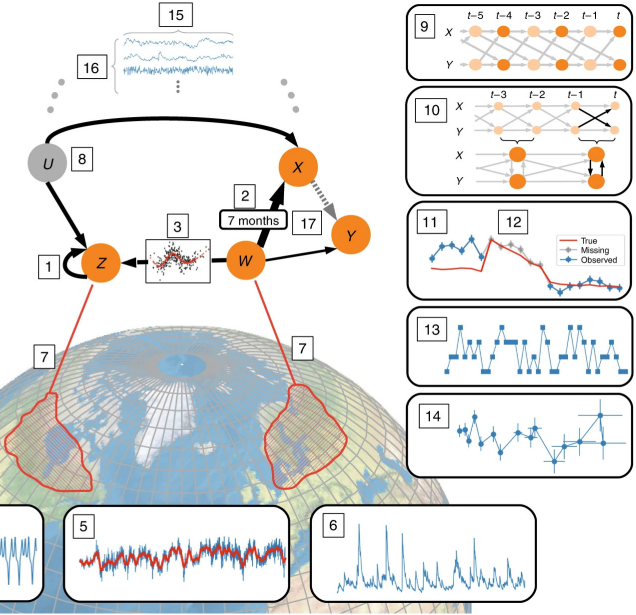
causal entities

Challenges

Process:
1 Autocorrelation
2 Time delays
3 Nonlinear dependencies
4 Chaotic state-dependence
5 Different time scales
6 Noise distributions

Data:
7 Variable extraction
8 Unobserved variables
9 Time subsampling
10 Time aggregation
11 Measurement errors
12 Selection bias
13 Discrete data
14 Dating uncertainties

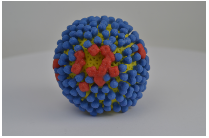
Computational/statistical:
15 Sample size
16 High dimensionality
17 Uncertainty estimation

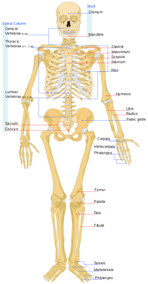


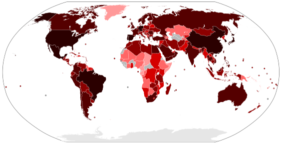
Runge et al. (2019). Inferring causation from time series in Earth system sciences. *Nature Communications*.

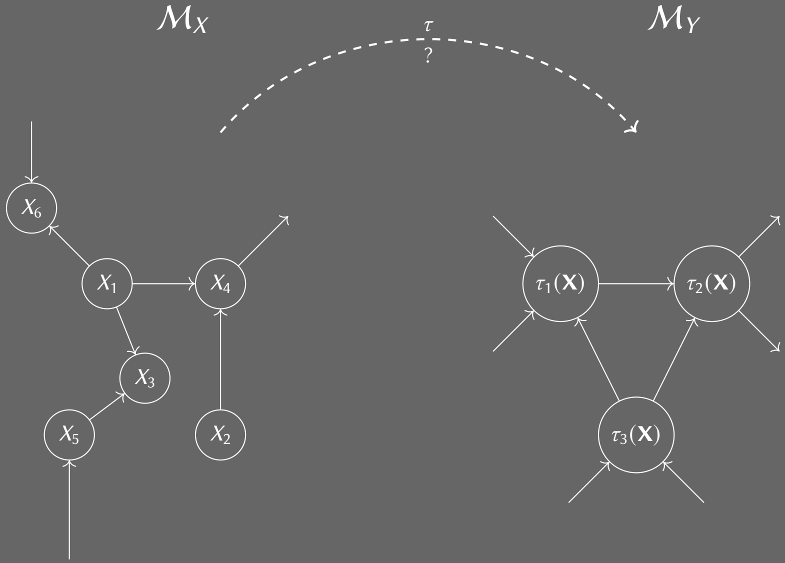
Variable Transformations may link Causal Reasoning at Different Scales

fine-grained ⟷ coarse-grained









$\mathcal{M}_X$
τ
?
$\mathcal{M}_Y$
X_6
X_1
X_4
X_3
X_5
X_2
$\tau_1(\mathbf{X})$
$\tau_2(\mathbf{X})$
$\tau_3(\mathbf{X})$

Causal Consistency of Structural Equation Models

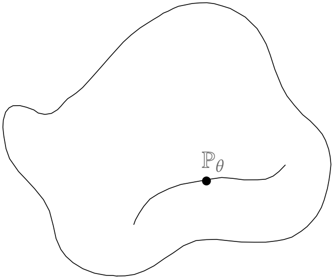
auai.org/uai2017/proceedings/papers/11.pdf

Paul Rubenstein, S Weichwald, S Bongers, JM Mooij,
D Janzing, M Grosse-Wentrup, B Schölkopf

Causal Models as Posets of Distributions

“Normal” Probabilistic Model:

$\mathcal{M}_X : \theta \mapsto \mathbb{P}_\theta$



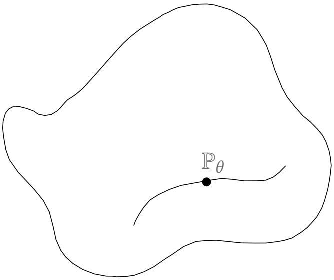
"Normal" Probabilistic Model:

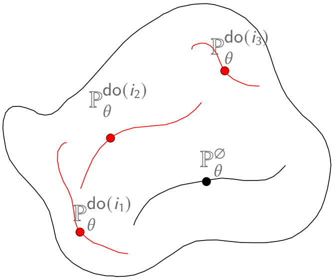
$$\mathcal{M}_X : \theta \mapsto \mathbb{P}_\theta$$

Causal Model:

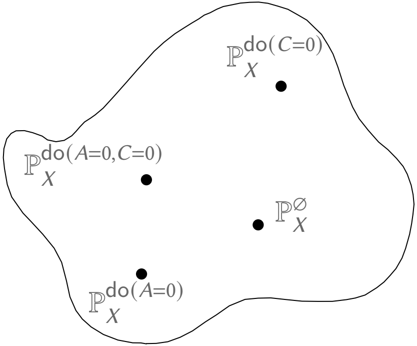
$$\mathcal{M}_X : \theta \mapsto \{\mathbb{P}_\theta^{\mathrm{do}(i)} \ : \ i \in \mathcal{I}_X\}$$

$\mathcal{I}_X$ is set of interventions.

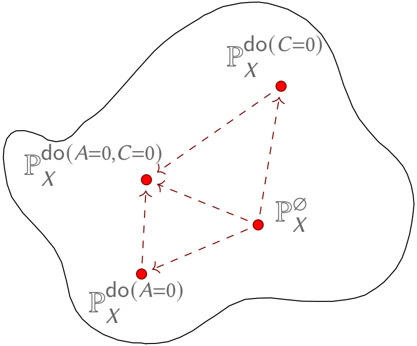




Causal Models

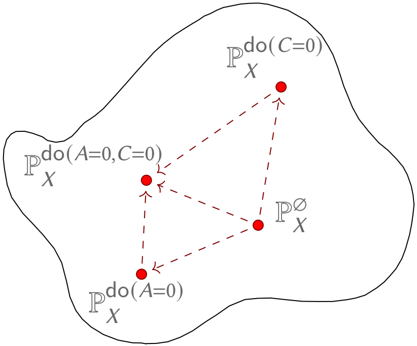


Causal Models



$\mathcal{I}_X$ has partial ordering structure

Causal Models



$\mathcal{I}_X$ has partial ordering structure

$\mathcal{M}_X$ implies the poset of distributions $\mathcal{P}_X := \left(\left\{\mathbb{P}_X^{\mathrm{do}(i)} \; : \; i \in \mathcal{I}_X\right\}, \leq_X\right)$

Structural Causal Models

$$\mathcal{M}_X = (\mathcal{S}_X, \mathcal{I}_X, \mathbb{P}_{E_X})$$

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observational

$$\mathbb{P}^{\varnothing}_{X_1} \sim \mathcal{N}(0, 1)$$

$$\mathbb{P}^{\varnothing}_{X_2} \sim \mathcal{N}(0, 2)$$

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observational	intervention on X_1
$\mathbb{P}^{\varnothing}_{X_1} \sim \mathcal{N}(0, 1)$	$\mathbb{P}^{\mathrm{do}(X_1=5)}_{X_1} \equiv 5$
$\mathbb{P}^{\varnothing}_{X_2} \sim \mathcal{N}(0, 2)$	$\mathbb{P}^{\mathrm{do}(X_1=5)}_{X_2} \sim \mathcal{N}(5, 1)$

Structural Causal Models

$$\mathcal{M}_X = (\mathcal{S}_X, \mathcal{I}_X, \mathbb{P}_{E_X})$$

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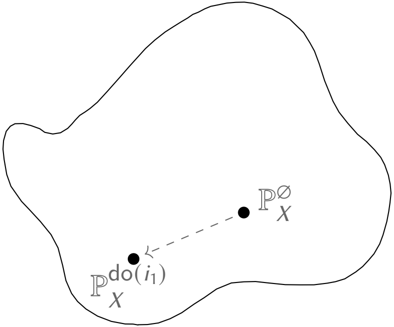
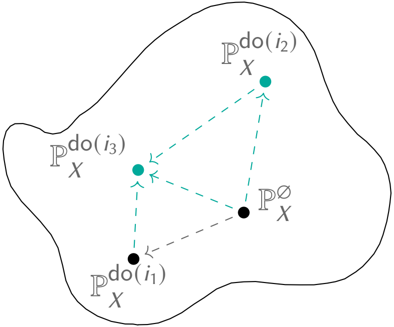
observational	intervention on X_1	intervention on X_2
$\mathbb{P}^{\emptyset}_{X_1} \sim \mathcal{N}(0,1)$	$\mathbb{P}^{\mathrm{do}(X_1=5)}_{X_1} \equiv 5$	$\mathbb{P}^{\mathrm{do}(X_2=3)}_{X_1} \sim \mathcal{N}(0,1)$
$\mathbb{P}^{\emptyset}_{X_2} \sim \mathcal{N}(0,2)$	$\mathbb{P}^{\mathrm{do}(X_1=5)}_{X_2} \sim \mathcal{N}(5,1)$	$\mathbb{P}^{\mathrm{do}(X_2=3)}_{X_2} \equiv 3$

causal discovery?

observations → causal model





$\left\{\mathbb{P}_X^{do(i)} : i \in \mathcal{I}_{sub} \subsetneq \mathcal{I}_X\right\}$

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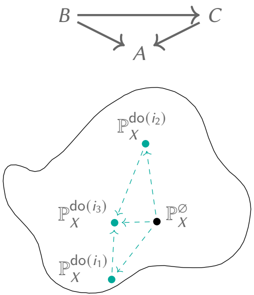
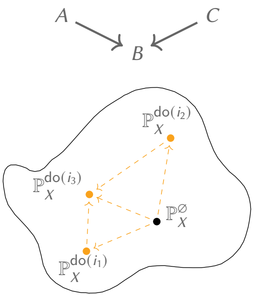
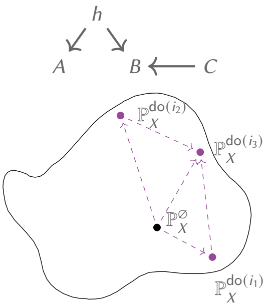


$$A = {}^1\!/\!_2 B - {}^1\!/\!_2 C + \sqrt{{}^3\!/\!_2} N_A$$
$$B = \sqrt{3} N_B$$
$$C = {}^1\!/\!_3 B + \sqrt{{}^2\!/\!_3} N_C$$



$$A = \sqrt{2} N_A$$
$$B = {}^1\!/\!_2 A + C + \sqrt{{}^3\!/\!_2} N_B$$
$$C = N_C$$



$$A = h + N_A$$
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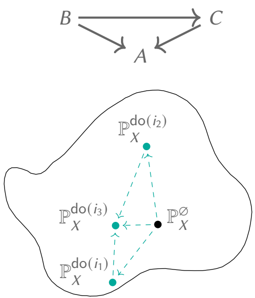

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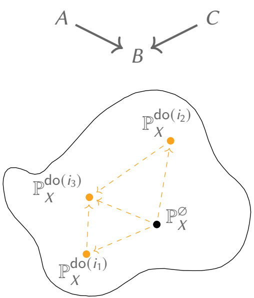

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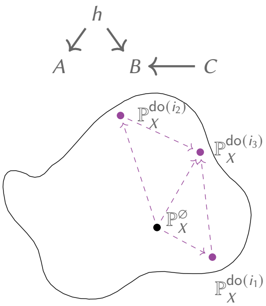

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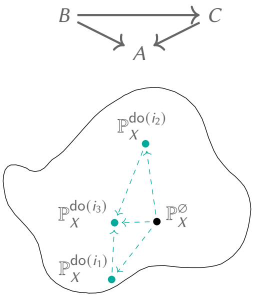
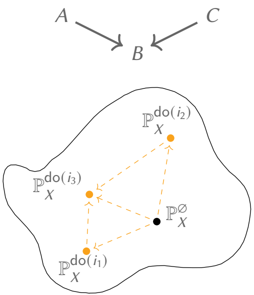
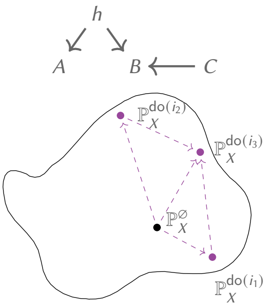
$\rightsquigarrow$ 3 models inducing the same observational yet different interventional distributions







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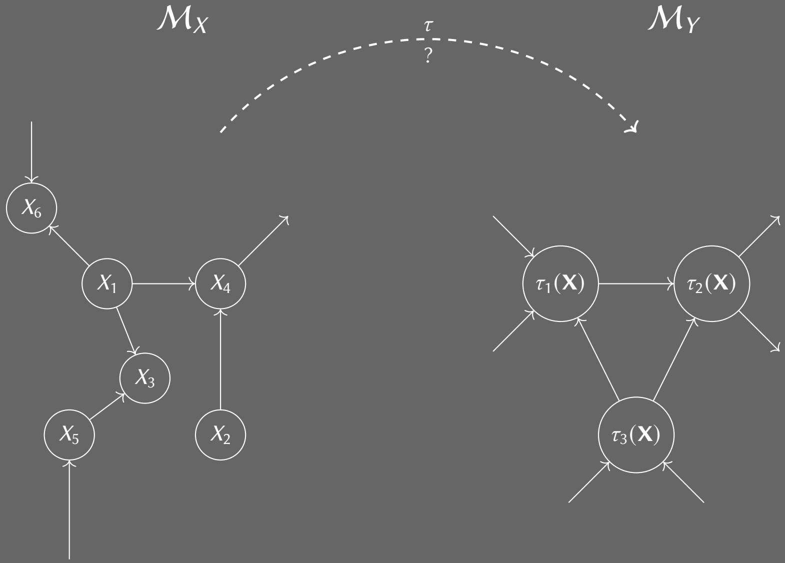
⇝ 3 models inducing the same observational yet different interventional distributions

fitting observational data well is not enough ↯



$\mathcal{M}_X$
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NEEDED: ALGEBRA OF DOING

Available: algebra of **seeing**

e.g., What is the chance it rained if we **see** the grass wet?

$$P\,(rain \mid wet) = ? \qquad \{= P\,(wet \mid rain)\frac{P(rain)}{P(wet)}\}$$

Needed: algebra of **doing**

e.g., What is the chance it rained if we **make** the grass wet?

$$P\,(rain \mid do(wet)) = ? \quad \{= P\,(rain)\}$$

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Pearl (2009). Causality: Models, Reasoning, and Inference.

The Three Layer Causal Hierarchy

Level (Symbol)	Typical Activity	Typical Questions	Examples
1. Association $P(y\|x)$	Seeing	What is? How would seeing X change my belief in Y?	What does a symptom tell me about a disease? What does a survey tell us about the election results?
2. Intervention $P(y\|do(x), z)$	Doing Intervening	What if? What if I do X?	What if I take aspirin, will my headache be cured? What if we ban cigarettes?
3. Counterfactuals $P(y_x\|x', y')$	Imagining, Retrospection	Why? Was it X that caused Y? What if I had acted differently?	Was it the aspirin that stopped my headache? Would Kennedy be alive had Oswald not shot him? What if I had not been smoking the past 2 years?

Pearl (2018). Theoretical Impediments to Machine Learning – With Seven Sparks from the Causal Revolution. *WSDM 2018.*

The Three Layer Causal Hierarchy

presupposes that X and Y are *the right* variables

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$X \sim \mathbb{P}_X$ an r.v. in $\mathcal{X} \implies \tau(X) \sim \mathbb{P}_{\tau(X)}$ is an r.v. in $\mathcal{Y}$

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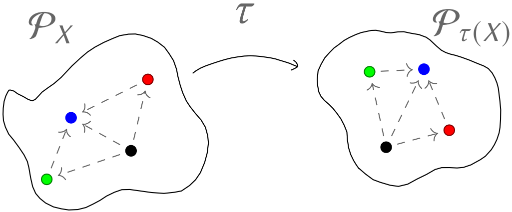

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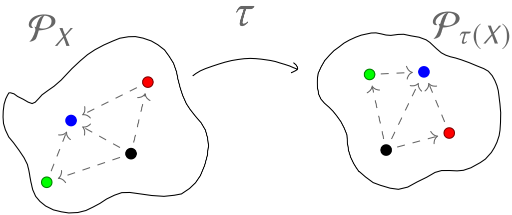

Does there exist an SCM $\mathcal{M}_Y$ with $\mathcal{P}_Y = \mathcal{P}_{\tau(X)}$?

If so, then $\mathcal{M}_Y$ will agree with our observations of $\mathcal{M}_X$ via τ.

What can go wrong?

$\mathcal{M}_X$: $X_1 \rightarrow X_3 \leftarrow X_2$

$$\mathcal{S}_X = \begin{cases} X_1 = E_1 \\ X_2 = E_2 \\ X_3 = X_1 + X_2 + E_3 \end{cases}$$

$E_2 = 1$; E_1, E_3 arbitrary

$$\mathcal{I}_X = \begin{cases} \mathrm{do}(\emptyset) \\ \mathrm{do}(X_1 = 0) \\ \mathrm{do}(X_1 = 0, X_2 = 0) \end{cases}$$

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$\mathcal{M}_X$: $X_1 \rightarrow X_3 \leftarrow X_2$

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$\mathcal{M}_X$: $X_1 \rightarrow X_3 \leftarrow X_2$

$\mathcal{M}_Y$: $Y_1 = X_1 + X_2 \rightarrow Y_2 = X_3$

$$\mathcal{S}_X = \begin{cases} X_1 = E_1 \\ X_2 = E_2 \\ X_3 = X_1 + X_2 + E_3 \end{cases}$$

$$\mathcal{S}_Y = \begin{cases} Y_1 = E_1 + E_2 \\ Y_2 = Y_1 + E_3 \end{cases}$$

$E_2 = 1$; E_1, E_3 arbitrary

$$\mathcal{I}_X = \begin{cases} \mathrm{do}(\emptyset) \\ \mathrm{do}(X_1 = 0) \\ \mathrm{do}(X_1 = 0, X_2 = 0) \end{cases}$$

What can go wrong?

$$\mathcal{M}_X : \quad X_1 \rightarrow X_3 \leftarrow X_2$$

$$\mathcal{M}_Y : \quad Y_1 = X_1 + X_2 \;\longrightarrow\; Y_2 = X_3$$

$$\mathcal{S}_X = \begin{cases} X_1 = E_1 \\ X_2 = E_2 \\ X_3 = X_1 + X_2 + E_3 \end{cases}$$

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E_1, E_2, E_3 as before

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What can go wrong?

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$$\mathcal{I}_Y = \begin{cases} \mathrm{do}(\emptyset) \\ \\ \\ \end{cases}$$

What can go wrong?

$$\mathcal{M}_X : \quad X_1 \rightarrow X_3 \leftarrow X_2$$

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E_1, E_2, E_3 as before

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What can go wrong?

$\mathcal{M}_X$: $X_1 \rightarrow X_3 \leftarrow X_2$

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What can go wrong?

$$\mathcal{M}_X : \quad \begin{array}{ccc} X_1 & & X_2 \\ & \searrow \quad \swarrow & \\ & X_3 & \end{array}$$

$$\mathcal{M}_Y : \quad \begin{array}{c} Y_1 = X_1 + X_2 \\ \downarrow \\ Y_2 = X_3 \end{array}$$

What can go wrong?

$\mathcal{M}_X$: $X_1 \searrow X_3 \swarrow X_2$ (i.e. $X_1 \to X_3 \leftarrow X_2$)

$\mathcal{M}_Y$: $Y_1 = X_1 + X_2 \to Y_2 = X_3$

$$\mathbb{P}_X \longrightarrow \mathbb{P}_X^{\text{do}(X_1=0)} \longrightarrow \mathbb{P}_X^{\text{do}(X_1=0, X_2=0)}$$

What can go wrong?

$\mathcal{M}_X$:

$$X_1 \searrow X_3 \swarrow X_2$$

$\mathcal{M}_Y$:

$$Y_1 = X_1 + X_2 \downarrow Y_2 = X_3$$

$$\mathbb{P}_X \longrightarrow \mathbb{P}_X^{do(X_1=0)} \longrightarrow \mathbb{P}_X^{do(X_1=0, X_2=0)}$$

$$\mathbb{P}_X \xrightarrow{\tau} \mathbb{P}_Y$$

What can go wrong?

$\mathcal{M}_X$: $X_1 \rightarrow X_3 \leftarrow X_2$

$\mathcal{M}_Y$: $Y_1 = X_1 + X_2 \rightarrow Y_2 = X_3$

$\mathbb{P}_X \longrightarrow \mathbb{P}_X^{do(X_1=0)} \longrightarrow \mathbb{P}_X^{do(X_1=0, X_2=0)}$

τ: $\mathbb{P}_X \rightarrow \mathbb{P}_Y$

τ: $\mathbb{P}_X^{do(X_1=0)} \rightarrow \mathbb{P}_Y^{do(Y_1=1)}$

What can go wrong?

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$$\mathbb{P}_X \longrightarrow \mathbb{P}_X^{do(X_1=0)} \longrightarrow \mathbb{P}_X^{do(X_1=0, X_2=0)}$$

$$\downarrow \tau \qquad \downarrow \tau \qquad \downarrow \tau$$

$$\mathbb{P}_Y \qquad \mathbb{P}_Y^{do(Y_1=1)} \qquad \mathbb{P}_Y^{do(Y_1=0)}$$

What can go wrong?

$\mathcal{M}_X :$ $X_1 \rightarrow X_3 \leftarrow X_2$

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$$
\begin{array}{ccccc}
\mathbb{P}_X & \longrightarrow & \mathbb{P}_X^{do(X_1=0)} & \longrightarrow & \mathbb{P}_X^{do(X_1=0, X_2=0)} \\
\downarrow \tau & & \downarrow \tau & & \downarrow \tau \\
\mathbb{P}_Y & \longrightarrow & \mathbb{P}_Y^{do(Y_1=1)} & & \mathbb{P}_Y^{do(Y_1=0)}
\end{array}
$$

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$$
\begin{array}{ccccc}
\mathbb{P}_X & \longrightarrow & \mathbb{P}_X^{\text{do}(X_1=0)} & \longrightarrow & \mathbb{P}_X^{\text{do}(X_1=0,X_2=0)} \\
\downarrow \tau & & \downarrow \tau & & \downarrow \tau \\
\mathbb{P}_Y & \longrightarrow & \mathbb{P}_Y^{\text{do}(Y_1=1)} & \times & \mathbb{P}_Y^{\text{do}(Y_1=0)}
\end{array}
$$

What can go wrong?

$$\mathcal{M}_X : \quad X_1 \searrow X_3 \swarrow X_2$$

$$\mathcal{M}_Y : \quad Y_1 = X_1 + X_2 \longrightarrow Y_2 = X_3$$

$$\begin{array}{ccccc}
\mathbb{P}_X & \longrightarrow & \mathbb{P}_X^{do(X_1=0)} & \longrightarrow & \mathbb{P}_X^{do(X_1=0, X_2=0)} \\
\downarrow \tau & & \downarrow \tau & & \downarrow \tau \\
\mathbb{P}_Y & \longrightarrow & \mathbb{P}_Y^{do(Y_1=1)} & \times & \mathbb{P}_Y^{do(Y_1=0)}
\end{array}$$

$$\mathbb{P}_Y \longrightarrow \mathbb{P}_Y^{do(Y_1=0)}$$

Exact Transformations ensure Causally Consistent SCMs

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Let $\mathcal{M}_X = (\mathcal{S}_X, \mathcal{I}_X, \mathbb{P}_{E_X})$ be an SCM over variables $X = (X_i : i \in \mathbb{I}_X)$ with

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Let $\mathcal{M}_Y = (\mathcal{S}_Y, \mathcal{I}_Y, \mathbb{P}_{E_Y})$ be another SCM and $\tau : \mathcal{X} \to \mathcal{Y}$.

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$\mathcal{M}_Y$ is an exact τ-transformation of $\mathcal{M}_X$ if

$$\mathbb{P}^i_{\tau(X)} = \mathbb{P}_Y^{\mathrm{do}(\omega(i))} \qquad \forall i \in \mathcal{I}_X$$

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$\mathcal{M}_Y$ is an exact τ-transformation of $\mathcal{M}_X$ if

$$\mathbb{P}^{i}_{\tau(X)} = \mathbb{P}^{\mathrm{do}(\omega(i))}_{Y} \qquad \forall i \in \mathcal{I}_X$$

for a surjective order-preserving map $\omega : \mathcal{I}_X \to \mathcal{I}_Y$.

Exact Transformations ensure Causally Consistent SCMs

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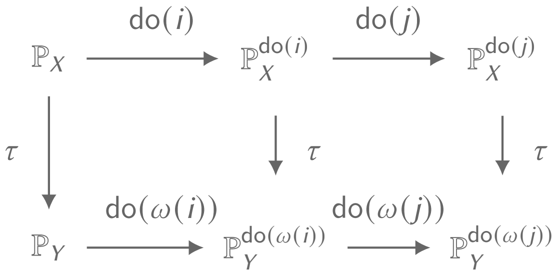
for a surjective order-preserving map $\omega : \mathcal{I}_X \to \mathcal{I}_Y$.

$$\implies \mathcal{M}_X \text{ and } \mathcal{M}_Y \text{ are causally consistent}$$

Causal Consistency



Elementary Properties of Exact Transformations

Lemma

The identity mapping is an exact transformation.

$$\mathcal{M}_X \xrightarrow{\text{id}} \mathcal{M}_X$$

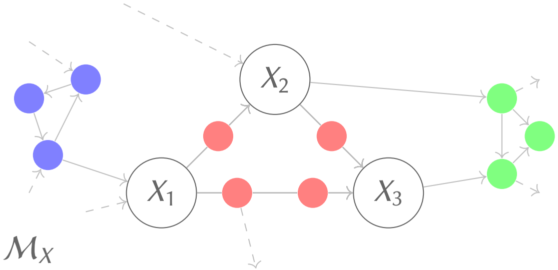
Lemma

Exact transformations are transitively closed.

$$\mathcal{M}_X \xrightarrow{\tau_1} \mathcal{M}_Y \xrightarrow{\tau_2} \mathcal{M}_Z, \qquad \mathcal{M}_X \xrightarrow{\tau_2 \circ \tau_1} \mathcal{M}_Z$$

Few Transformations yield Causally Consistent Representations ↯

- Marginalisation of variables



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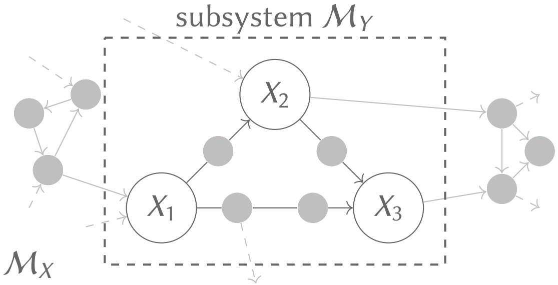
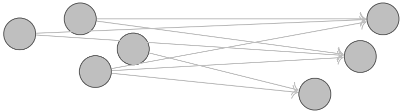


Few Transformations yield Causally Consistent Representations ↯

- Marginalisation of variables
- Micro- to macro-level and aggregate features

$\mathcal{M}_X$:



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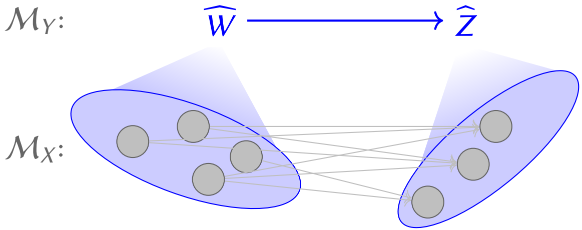

Few Transformations yield Causally Consistent Representations ⚡

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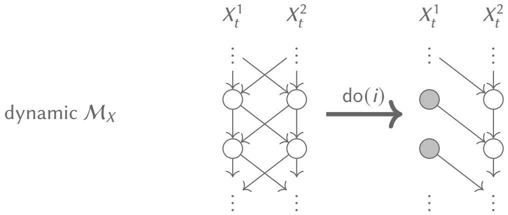

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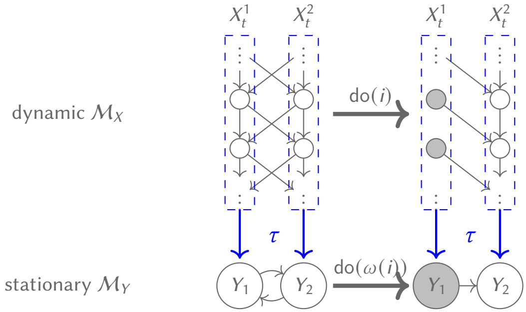

Causal Models under Variable Transformations

Challenges for *Causally Consistent* Representation Learning

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- Variable Transformations may break Causal Reasoning ↯

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